Co-amenable Quantum Homogeneous Spaces of Compact Kac Quantum Groups

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Amenable groups

Definition

A locally compact group G is *amenable* if there exists a state m on $L^\infty(G)$ which is invariant under the left translation action: for all $s \in G$ and $x \in L^\infty(G)$, $m(s \cdot x) = m(x)$.

Theorem (Leptin 1968)

A locally compact group G is amenable if and only if $A(G)$ admits a bounded approximate identity.

Let G be an amenable group. Then

- The C^*-algebra $C_r^*(G)$ is nuclear,
- the von Neumann algebra $\mathrm{vN}(G)$ is injective.

The converse is true if G is a discrete group.

Locally compact quantum groups

Von Neumann algebraic version

A von Neumann algebraic locally compact quantum group is a quadruple $(M, \Delta, \varphi, \psi)$, where

- M is a von Neumann algebra,
- $\Delta\colon M \to M \bar{\otimes} M$ is a normal, unital, injective $*$-homomorphism satisfies the coassociativity condition, i.e. $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$,
- φ and ψ are left and right invariant normal semifinite faithful weights on M respectively,

$$(\mathrm{id} \otimes \varphi)\Delta = \mathbb{1}\varphi$$
$$(\psi \otimes \mathrm{id})\Delta = \mathbb{1}\psi$$

φ and ψ are called left and right Haar weights respectively.

Reduced C*-algebraic version

A reduced C*-algebraic locally compact quantum group is a quadruple $(A, \Delta, \varphi, \psi)$, where A is a C*-algebra with a coassociative map

$$\Delta : A \to \mathrm{M}(A \otimes A), \quad \text{s.t} \quad (A \otimes \mathbb{1})\Delta(A) = A \otimes A = (\mathbb{1} \otimes A)\Delta(A),$$

and φ and ψ are left and right invariant faithful, proper, KMS-weights on A respectively.

Universal C*-algebraic version

A universal C*-algebraic locally compact quantum group is a quadruple $(A^u, \Delta^u, \varphi^u, \psi^u)$, where A^u is a C*-algebra with a coassociative map

$$\Delta^u : A^u \to \mathrm{M}(A^u \otimes A^u), \quad \text{s.t} \quad (A^u \otimes \mathbb{1})\Delta^u(A^u) = A^u \otimes A^u = (\mathbb{1} \otimes A^u)\Delta(A^u)$$

and φ^u and ψ^u are left and right invariant ~~faithful~~, proper, KMS-weights on A^u respectively.

All of these different aspects study the same object denoted by $\mathbb{G}$.

$$A \longrightarrow C_0(\mathbb{G}), \quad A^u \longrightarrow C_0^u(\mathbb{G}), \quad M \longrightarrow L^\infty(\mathbb{G}).$$

- Universal C^*-algebraic quantum group always has a co-unit, i.e. a $*$-homomorphism $\epsilon : C_0^u(\mathbb{G}) \to \mathbb{C}$ such that

$$(\epsilon \otimes \mathrm{id})\Delta^u = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta^u.$$

- There exists a surjective $*$-homomorphism $\Lambda : C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$ which is injective if and only if $C_0(\mathbb{G})$ has a co-unit.

Every locally compact quantum group $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta_{\mathbb{G}}, \varphi_{\mathbb{G}}, \psi_{\mathbb{G}})$, admits a dual quantum group $\widehat{\mathbb{G}} = (L^\infty(\widehat{\mathbb{G}}), \Delta_{\widehat{\mathbb{G}}}, \varphi_{\widehat{\mathbb{G}}}, \psi_{\widehat{\mathbb{G}}})$ such that

$$\mathbb{G} \cong \widehat{\widehat{\mathbb{G}}}.$$

Example

Let G be a locally compact group. Let $\mathbb{G} = (L^\infty(G), \Delta, \varphi, \psi)$, then its dual quantum group is $\widehat{\mathbb{G}} = (\mathrm{vN}(G), \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$, where $\widehat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$ and $\widehat{\varphi} = \widehat{\psi}$ is the Plancharel weight.

- The preadjoint of Δ induces a multiplication on $L^1(\mathbb{G}) = L^\infty(\mathbb{G})_*$,

$$* : L^1(\mathbb{G}) \otimes L^1(\mathbb{G}) \to L^1(\mathbb{G}), \quad (f \otimes g) \mapsto f * g = (f \otimes g) \circ \Delta, \quad f, g \in L^1(\mathbb{G}).$$

- There are canonical left and right $L^1(\mathbb{G})$-module structure on $L^\infty(\mathbb{G})$,

$$f * x = (\mathrm{id} \otimes f)\Delta(x), \quad x * f = (f \otimes \mathrm{id})\Delta(x), \quad f \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G}).$$

Definition

A locally compact quantum group $\mathbb{G}$ is called

- *amenable* if $L^\infty(\mathbb{G})$ admits a left invariant mean, i.e. a state $m \in L^\infty(\mathbb{G})^*$ s.t. $m(f * x) = f(1)m(x)$, $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$.
- *co-amenable* if $L^1(\mathbb{G})$ has a bounded approximate identity.

Question: Is amenability of $\mathbb{G}$ equivalent to the coamenability of $\widehat{\mathbb{G}}$?

Co-amenability and Amenability

Theorem (Bedos-Tuset)

Let $\mathbb{G}$ be a locally compact quantum group. The following are equivalent:

1. *The Banach algebra $L^1(\mathbb{G})$ has a bounded approximate identity,*
2. *There is a state $\epsilon \in C_0(\mathbb{G})^*$ such that $(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id}$,*
3. *$C_0^u(\mathbb{G}) = C_0(\mathbb{G})$, that is the map $\Lambda : C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$ is injective.*

Theorem (Bedos-Tuset)

If a locally compact quantum group $\mathbb{G}$ is coamenable, then its dual quantum group $\widehat{\mathbb{G}}$ is amenable.

Theorem (Bedos-Tuset)

Suppose that $\mathbb{G}$ is an amenable quantum group. Then $C_0^u(\widehat{\mathbb{G}})$ and $C_0(\widehat{\mathbb{G}})$ are nuclear C^*-algebras and $L^\infty(\widehat{\mathbb{G}})$ is an injective von Neumann algebra.

$\mathbb{G}$-injectivity

- If $\widehat{\mathbb{G}}$ is an amenable locally compact group, then $L^\infty(\mathbb{G})$ is injective.

Definition

Let $\mathbb{G}$ be a locally compact quantum group, and let M be a $\mathbb{G}$-von Neumann algebra, i.e. there exists an action $\alpha : \mathbb{G} \curvearrowright M$. We say M is $\mathbb{G}$-injective von Neumann algebra if for any $\mathbb{G}$-von Neumann algebras N_1, N_2, every completely isometric $\mathbb{G}$-equivariant $\Psi : N_1 \to N_2$, and every ucp $\mathbb{G}$-equivariant map $\Phi : N_1 \to M$, there exists a ucp $\mathbb{G}$-equivariant map $\tilde{\Phi} : N_2 \to M$ such that $\tilde{\Phi} \circ \Psi = \Phi$.

Theorem (Crann-2017)

A locally compact quantum group $\widehat{\mathbb{G}}$ is amenable if and only if $L^\infty(\mathbb{G})$ is $\mathbb{G}$-injective.

A locally compact quantum group $\mathbb{G}$ is

- *discrete* if $L^1(\mathbb{G}) = L^\infty(\mathbb{G})_*$ is unital, denoted by $l^1(\mathbb{G}) = l^\infty(\mathbb{G})_*$.
- *compact* if $C_0(\mathbb{G})$ is unital, denoted by $C(\mathbb{G})$ and Haar weights are finite.
- *compact Kac* if $\varphi^{\mathbb{G}}$ is a tracial state.
- *discrete Kac* if $\varphi^{\mathbb{G}} = \psi^{\mathbb{G}}$.

Theorem (Ruan-1996)

Let $\mathbb{G}$ be a discrete Kac quantum group and let $\widehat{\mathbb{G}}$ be its compact Kac dual quantum group. Then the following are equivalent:

1. *$\mathbb{G}$ is amenable,*
2. *$L^\infty(\widehat{\mathbb{G}})$ is injective,*
3. *$C_0(\widehat{\mathbb{G}})$ and $C_0^u(\widehat{\mathbb{G}})$ are nuclear C^*-algebras,*
4. *$\widehat{\mathbb{G}}$ is coamenable.*

Blanchard-Vaes (2002), Tomatsu (2006), and Crann (2017) have shown that for a discrete quantum group $\mathbb{G}$,

$$\mathbb{G} \text{ is amenable if and only if } \widehat{\mathbb{G}} \text{ is coamenable}$$

Relative amenability

Definition

Let G be a locally compact group. The closed subgroup $H < G$ is called *relatively amenable* if there is a left H-invariant mean on $C_b^{ru}(G)$.

Let G be a locally compact group. The closed subgroup $H < G$ is

- *amenable* iff there is a G-equivariant conditional expectation $E : L^\infty(G) \to L^\infty(G/H)$.
- *relatively amenable* iff there is a G-equivariant positive norm one linear map $E : L^\infty(G) \to L^\infty(G/H)$.

Caprace-Monod showed that for a large class of locally compact groups (containing all discrete groups and all groups amenable at infinity) every relatively amenable subgroup is amenable.

Question: Is there any relative amenable subgroup that is not amenable?

Closed quantum subgroups

Definition

We say $\mathbb{H}$ is a *closed quantum subgroup* of $\mathbb{G}$ if

- (Woronowicz) there is a surjective $*$-homomorphism

$$\pi : \mathrm{C}_0^u(\mathbb{G}) \to \mathrm{C}_0^u(\mathbb{H}) \quad \text{s.t} \quad (\pi \otimes \pi) \circ \Delta_{\mathbb{G}}^u = \Delta_{\mathbb{H}}^u \circ \pi.$$

- (Vaes) there exists an injective normal unital $*$-homomorphism

$$\gamma : L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}}) \quad \text{s.t} \quad (\gamma \otimes \gamma) \circ \Delta_{\widehat{\mathbb{H}}} = \Delta_{\widehat{\mathbb{G}}} \circ \gamma.$$

Let $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$. Then $\mathbb{H}$ acts on $L^\infty(\mathbb{G})$ on the right

$$\alpha : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{H}).$$

The fixed point algebra of α is denoted by

$$L^\infty(\mathbb{G}/\mathbb{H}) = \left\{ x \in L^\infty(\mathbb{G}) \,\middle|\, \alpha(x) = x \otimes \mathbb{1} \right\}$$

Coideal

Let $\mathbb{G}$ be a locally compact quantum group. A von Neumann subalgebra $M \subseteq L^\infty(\mathbb{G})$ is called:

- (left) coideal if $\Delta(M) \subseteq L^\infty(\mathbb{G})\overline{\otimes}M$,
- invariant if $\Delta(M) \subseteq M\overline{\otimes}M$.

There is a one-one correspondance between (left) coideals of $\mathbb{G}$ and (left) coideals of $\widehat{\mathbb{G}}$.

$$M \text{ coideal in } L^\infty(\mathbb{G}) \quad \Longleftrightarrow \quad \tilde{M} = M' \cap L^\infty(\widehat{\mathbb{G}}) \text{ coideal in } \; L^\infty(\widehat{\mathbb{G}}).$$

Example

Let $\mathbb{H}$ be a closed quantum group of $\mathbb{G}$. Then $M = L^\infty(\mathbb{G}/\mathbb{H})$ is a coideal in $\mathbb{G}$. Moreover its codual $\tilde{M} = L^\infty(\widehat{\mathbb{H}})$.

Relative amenability

Definition (Kalantar-Kasprzak-Skalski-Vergnioux 2022)

Let $\mathbb{G}$ be a discrete quantum group, and let $\mathbb{H}$ be a quantum subgroup of $\mathbb{G}$. We say $\mathbb{H}$ is relatively amenable in $\mathbb{G}$ if there is a $\mathbb{H}$-invariant mean on $\ell^\infty(\mathbb{G})$.

Let $\mathbb{G}$ be a discrete quantum group. A coideal $M \subseteq \ell^\infty(\mathbb{G})$ is said to be

- amenable coideal in $\ell^\infty(\mathbb{G})$ if there is a $\mathbb{G}$-equivariant conditional expectation $E : \ell^\infty(\mathbb{G}) \to M$.
- relative amenable coideal in $\ell^\infty(\mathbb{G})$ if there is a ucp $\mathbb{G}$-equivariant map $E : \ell^\infty(\mathbb{G}) \to M$.

Relative amenability and Amenability

Theorem (Kalantar-Kasprzak-Skalski-Vergnioux 2022)

Let $\mathbb{G}$ be a discrete quantum group and $\mathbb{H}$ be a quantum subgroup of $\mathbb{G}$. The following are equivalent:

- $\mathbb{H}$ *is relatively amenable in* $\mathbb{G}$*,*
- $\ell^\infty(\mathbb{G}/\mathbb{H})$ *is relatively amenable coideal in* $\ell^\infty(\mathbb{G})$*,*
- $\mathbb{H}$ *is amenable,*
- $\ell^\infty(\mathbb{G}/\mathbb{H})$ *is amenable coideal in* $\ell^\infty(\mathbb{G})$*.*

Note that the trivial coideal $\mathbb{C}$ is (relative) amenable if and only if the discrete quantum group $\mathbb{G}$ is amenable.

Question: Let M be a coideal of $\ell^\infty(\mathbb{G})$. Under which condition relative amenability of a coideal M implies amenability of M in $\ell^\infty(\mathbb{G})$?

Coamenability of quantum homogeneous spaces

Let $\mathbb{G}$ be a compact quantum group and let $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$ realized by $\pi \in \mathrm{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$. Denote

$$C^u(\mathbb{G}/\mathbb{H}) = \{x \in C^u(\mathbb{G}) \,|\, (id \otimes \pi)\Delta^u(x) = x \otimes 1\}, \quad C(\mathbb{G}/\mathbb{H}) = \Lambda_{\mathbb{G}}(C^u(\mathbb{G}/\mathbb{H})).$$

Definition (Kalantar-Kasprzak-Skalski-Vergnioux 2022)

We say that $\mathbb{G}/\mathbb{H}$ is a coamenable quantum quotient of $\mathbb{G}$ if the restriction $\varepsilon : C^u(\mathbb{G}/\mathbb{H}) \to \mathbb{C}$ admits a reduced version, i.e. there exists $\varepsilon^r : C(\mathbb{G}/\mathbb{H}) \to \mathbb{C}$ satisfying $\varepsilon^r \circ \Lambda_{\mathbb{G}}(C^u(\mathbb{G}/\mathbb{H})) = \varepsilon|_{C^u(\mathbb{G}/\mathbb{H})}$.

Theorem (Kalantar-Kasprzak-Skalski-Vergnioux 2022)

The quantum quotient $\mathbb{G}/\mathbb{H}$ is co-amenable if and only if $\pi \in \mathrm{Mor}(C^u(\mathbb{G}), C^u(\mathbb{H}))$ admits a reduced version, that is there exists $\pi^r \in \mathrm{Mor}(C(\mathbb{G}), C(\mathbb{H}))$ such that $\pi^r \circ \Lambda_{\mathbb{G}} = \Lambda_{\mathbb{H}} \circ \pi$.

Reminder. A LCQG $\mathbb{G}$ is coamenable if $\varepsilon \in C_0^u(\mathbb{G})^*$ admits a reduced version $\varepsilon^r \in C_0(\mathbb{G})^*$.

Coamenability of quantum homogeneous spaces

Theorem (Kalantar-Kasprzak-Skalski-Vergnioux 2022)

Let $\mathbb{G}$ be a compact quantum group and let $\mathbb{H} \leq \mathbb{G}$ be a normal quantum subgroup. Then the normal Baaj-Vaes subalgebra $\ell^\infty(\widehat{\mathbb{H}}) \subseteq \ell^\infty(\widehat{\mathbb{G}})$ is relatively amenable iff $\mathbb{G}/\mathbb{H}$ is a co-amenable quotient.

Proof.

- The quotient $\mathbb{G}/\mathbb{H}$ viewed as a compact quantum group denoted by $\mathbb{L}$,
- $\ell^\infty(\widehat{\mathbb{H}}) = \ell^\infty(\widehat{\mathbb{G}}/\widehat{\mathbb{L}})$ is relative amenable iff $\widehat{\mathbb{L}}$ is amenable,
- $\widehat{\mathbb{L}}$ is amenable iff $\mathbb{G}/\mathbb{H} = \mathbb{L}$ is coamenable.

Question. What is the relation between (relative) amenability of $\ell^\infty(\widehat{\mathbb{H}})$ in $\ell^\infty(\widehat{\mathbb{G}})$ and the co-amenability of the quotient $\mathbb{G}/\mathbb{H}$ in general for non-normal quantum subgroups $\mathbb{H} \leq \mathbb{G}$?

Let $\alpha : M \to L^\infty(\mathbb{G})\overline{\otimes}M$ be an action of $\mathbb{G}$ on M. For every normal semifinite faithful weight θ on M, there exists a unitary $U_\theta \in L^\infty(\mathbb{G})\overline{\otimes}B(H_\theta)$ such that

$$\alpha(x) = U_\theta^*(1 \otimes x)U_\theta.$$

Then U_θ implements an action of $\mathbb{G}^{op}$ on $M' \subseteq B(H_\theta)$,

$$\alpha^{op} : M' \to L^\infty(\mathbb{G}^{op})\overline{\otimes}M', \quad \alpha^{op}(x) = U_\theta(1 \otimes x)U_\theta^*.$$

Proposition (Moakhar 2018)

Let $\mathbb{G}$ be a compact quantum group of Kac type, and let $\mathbb{H} \leq \mathbb{G}$. Then the following are equivalent:

1. *There is a $\widehat{\mathbb{G}}$-equivariant conditional expectation $E : \ell^\infty(\widehat{\mathbb{G}}) \to \ell^\infty(\widehat{\mathbb{H}})$,*
2. *$\widehat{\mathbb{G}} \ltimes \ell^\infty(\widehat{\mathbb{H}})$ is $\mathbb{G}^{op}$-injective,*
3. *$\widehat{\mathbb{G}} \ltimes \ell^\infty(\widehat{\mathbb{H}})$ is injective,*
4. *$L^\infty(\mathbb{G}/\mathbb{H})'$ is $\mathbb{G}^{op}$-injective,*
5. *$L^\infty(\mathbb{G}/\mathbb{H})'$ is injective.*
6. *$L^\infty(\mathbb{G}/\mathbb{H})$ is injective.*

$\mathbb{G}$-injectivity

- It is known that a von Neumann algebra $M \subseteq B(H)$ is injective iff M' is injective.
- Let $\mathbb{G}$ be a compact quantum group of Kac type, and let $\mathbb{H} \leq \mathbb{G}$. Does $\mathbb{G}^{op}$-injectivity of $L^\infty(\mathbb{G}/\mathbb{H})'$ imply $\mathbb{G}$-injectivity of $L^\infty(\mathbb{G}/\mathbb{H})$?

Definition

Let M be a $\mathbb{G}$-von Neumann algebra, i.e. there is an action $\alpha : \mathbb{G} \curvearrowright M$. We call M relatively $\mathbb{G}$-injective if there exists a ucp $\mathbb{G}$-equivariant map

$$\Psi : L^\infty(\mathbb{G})\overline{\otimes}M \to M, \quad \Psi \circ \alpha = \mathrm{id}_M.$$

Theorem (Crann-2017)

If M is an injective $\mathbb{G}$-von Neumann algebra that is relatively $\mathbb{G}$-injective, then M is $\mathbb{G}$-injective.

Theorem (Kalantar-K)

Let $\mathbb{G}$ be a compact Kac quantum group, and let $\mathbb{H} \leq \mathbb{G}$. If $L^\infty(\mathbb{G}/\mathbb{H})'$ is relatively $\mathbb{G}^{op}$-injective. Then $L^\infty(\mathbb{G}/\mathbb{H})$ is relatively $\mathbb{G}$-injective.

Proposition (Kalantar-K)

If M is a $\mathbb{G}$-injective von Neumann algebra. Then

$$CB_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}), M) = span\ CP_{L^1(\mathbb{G})}(L^\infty(\mathbb{G}), M).$$

Proposition (Kalantar-K)

Let $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$. Let $L^\infty(\mathbb{G}/\mathbb{H})$ be a $\mathbb{G}$-injective von Neumann algebra. Then for every completely bounded $\mathbb{G}$-equivariant map $\Phi : C(\mathbb{G}) \to L^\infty(\mathbb{G}/\mathbb{H})$, there exists a functional $\mu_\Phi \in C^u(\mathbb{G})^*$ such that $\Phi = (\mu_\Phi \otimes \mathrm{id})\Delta_r^{u,r}$.

Amenability implies coamenability

Theorem (Kalantar-K)

Let $\widehat{\mathbb{G}}$ be an exact discrete Kac quantum group. If there exists a $\widehat{\mathbb{G}}$-equivariant conditional expectation $E : l^\infty(\widehat{\mathbb{G}}) \to l^\infty(\widehat{\mathbb{H}})$. Then $\mathbb{G}/\mathbb{H}$ is coamenable.

Proof. Let $\alpha : \mathbb{G} \curvearrowright L^\infty(\mathbb{G}/\mathbb{H})$,

- Since $L^\infty(\mathbb{G}/\mathbb{H})$ is relatively $\mathbb{G}$-injective we have the ucp $\mathbb{G}$-equivariant left inverse of α, $\Phi : L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G}/\mathbb{H}) \to L^\infty(\mathbb{G}/\mathbb{H})$
- $L^\infty(\mathbb{G}/\mathbb{H})$ is injective, so $\iota : C(\mathbb{G}/\mathbb{H}) \to L^\infty(\mathbb{G}/\mathbb{H})$ is a nuclear map, there exists a net of finite rank ucp maps (Ψ_i) such that $\Psi_i \to \iota$,
- Let $\Phi_i := \Phi \circ (\mathrm{id} \otimes \Psi_i) \circ \alpha|_{C(\mathbb{G}/\mathbb{H})}$,
- There exists a net $(\mu_i) \subseteq C(\mathbb{G}/\mathbb{H})^*$ such that $\Phi_i = (\mathrm{id} \otimes \mu_i) \circ \alpha$,
- the weak*-cluster point of (μ_i) is the reduced counit on $C(\mathbb{G}/\mathbb{H})^*$.

Thanks for your attention!